Johannes Emmerling
Natural resources and environmental economics, TSE

Tutorial 5 - Exhaustible Resources - Optimality

TOPIC: Consider an economy disposing of two natural resources. The first one is a renewable resource available at each time as a constant flow $\bar{x} > 0$. The second one is an exhaustible resource available as a stock. Denote by S_t the remaining level of this stock at time t and by $\hat{S}$, the initial amount of the stock at time $t = 0$. There are no costs of exploitation of the two resources and they are perfectly substituable at the consumption stage. Hence, denoting by c_t the consumption level of the two resources at time t and by s_t the rate of extraction of the non renewable resource, one has at each time: $c_t = \bar{x} + s_t$.

The objective of the society is to maximize the following felicity function:

$$\mathcal{U} = \int_0^\infty 2c_t^{1/2} e^{-\rho t} dt \ ,$$

ρ being the social discount rate, $\rho > 0$.

Q1: Express the constrained optimization problem of the society. Denote by λ_t the costate variable associated with the non renewable stock dynamics equation. Using optimal control results, compute the first order necessary conditions for such a problem and make an economic comment about these conditions in the interior case (that is, when $s_t > 0$). Verify that λ_t should be a constant, you will denote by λ.

Q2: Compute the expression of s_t as both a function of time and the costate variable λ, a function denoted by $s(t, \lambda)$. Describe the variations of $s(t, \lambda)$ with respect to time and λ. Deduce and explain why the time duration of the exploitation of the non renewable resource should be finite.

Q3: Compute the expression of T, the termination date of the extraction of the non renewable stock as a function of λ. What is the condition over λ to get $T \geq 0$? Verify that T should be a decreasing function of λ.

Q4: Making use of the initial stock condition, compute a relation between the initial stock $\hat{S}$, T, $\bar{x}$ and λ.

Q5: Invert the relation giving T as a function of λ to obtain an expression of λ as a function of T. Inserting this last relation into the initial stock condition in order to eliminate λ, show that the stock condition may be expressed as a simpler relation between only T, $\bar{x}$ and $\hat{S}$. Check that this new equation admits only one solution $T^*(\hat{S}, \bar{x})$.

Q6: What can be said about the optimal exhaustion time of the non renewable resource if: (1) $\hat{S}$ is increased ? (2) $\bar{x}$ is increased ? Comment these results in terms of arbitrages between renewable and non renewable resources use along an optimal path.

Q4: Making use of the initial stock condition, compute a relation between the initial stock $\hat{S}$, T, $\bar{x}$ and λ.

Q5: Invert the relation giving T as a function of λ to obtain an expression of λ as a function of T. Inserting this last relation into the initial stock condition in order to eliminate λ, show that the stock condition may be expressed as a simpler relation between only T, $\bar{x}$ and $\hat{S}$. Check that this new equation admits only one solution $T^*(\hat{S}, \bar{x})$.

Q6: What can be said about the optimal exhaustion time of the non renewable resource if: (1) $\hat{S}$ is increased ? (2) $\bar{x}$ is increased ? Comment these results in terms of arbitrages between renewable and non renewable resources use along an optimal path.